EFFECT OF BLOWING ON THE RESISTANCE OF A SPHERE IN LAMINAR FLOW OF VISCOUS FLUID

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UDC 532.516

Using the method of matching asymptotic expansions [1-3], a stationary field of velocities is obtained in the vicinity of a sphere for Reynolds numbers R and εR computed from the blowing velocity and the fluid flow, respectively; they satisfy the relations $\varepsilon R^2 \ll 1$ and $\varepsilon R \ll 1$. It is also shown that for intensive blowing ($R \gg 1$), the resistive force is considerably smaller than that found by using the Stokes formula. For weak blowing the results are in good agreement with the solution of Oseen.

The motion of fluid is governed by the Navier–Stokes equations and the continuity equation which, as it is known [4], yield the following equation for the stream function:

$$\frac{R}{r^2\sin\theta}\left(\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial r}-\frac{\partial\Psi}{\partial r}\frac{\partial}{\partial\theta}+2\operatorname{ctg}\theta\frac{\partial\Psi}{\partial r}-\frac{2}{r}\frac{\partial\Psi}{\partial\theta}\right)D^2\Psi=D^4\Psi$$

$$\left(\begin{array}{l}D^2=\dfrac{\partial^2}{\partial r^2}+\dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta},\quad R=\dfrac{wa}{\nu}\\ v_r r^2\sin\theta=\partial\Psi/\partial\theta,\quad v_\theta r\sin\theta=-\partial\Psi/\partial r\end{array}\right) \tag{1}$$

In the above a is the sphere radius, ν is the kinematic viscosity, Ψ and r are the dimensionless stream functions and the distance to the sphere center, w is the fluid velocity at the sphere surface, v_r and v_θ are the components of the dimensionless velocity in the spherical coordinate system with the polar axis along the direction of u, the velocity of the flow far away from the sphere. To change over to dimensional quantities, one has to replace r, v_r, v_θ, Ψ by ar, wv_r, wv_θ, and $wa^2\Psi$, respectively.

The boundary conditions are

$$\Psi=-\cos\theta,\ \partial\Psi/\partial r=0\quad\text{for}\quad r=1$$

$$\Psi\to{}^1/_2\varepsilon r^2\sin^2\theta\quad(\varepsilon=u/w)\quad\text{for}\quad r\to\infty \tag{2}$$

A solution of Eq. (1) with the boundary conditions (2) can be sought in the form

$$\Psi(r,\theta)=-\cos\theta+\varepsilon\Psi_1(r,\theta)+\varepsilon^2\Psi_2(r,\theta)+\ldots$$

Equation (1) then yields

$$\frac{R}{r^2}\left(\frac{\partial}{\partial r}-\frac{2}{r}\right)D^2\Psi_1-D^4\Psi_1=0 \tag{3}$$

$$\frac{R}{r^2}\left(\frac{\partial}{\partial r}-\frac{2}{r}\right)D^2\Psi_2-D^4\Psi_2= \tag{4}$$
$$=\frac{R}{r^2\sin\theta}\left[\frac{\partial\Psi_1}{\partial r}\left(\frac{\partial}{\partial\theta}-2\operatorname{ctg}\theta\right)-\frac{\partial\Psi_1}{\partial\theta}\left(\frac{\partial}{\partial r}-\frac{2}{r}\right)\right]D^2\Psi_1$$

Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskai Fiziki, No. 3, pp. 110-114, May-June, 1972. Original article submitted December 17, 1971.

By setting

$$\Psi_1(r,\ \theta) = \psi_1(x)\ \sin^2\theta, \qquad \Psi_2(r,\ \theta) = \psi_2(x)\ \sin^2\theta\cos\theta\ (r = Rx)$$

one can arrive at the equations

$$\left(\frac{d^2}{dx^2} - \frac{2}{x^2} - \frac{1}{x^2}\frac{d}{dx} + \frac{2}{x^3}\right)\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)\psi_1 = 0 \tag{5}$$

$$\left(\frac{d^2}{dx^2} - \frac{6}{x^2} - \frac{1}{x^2}\frac{d}{dx} + \frac{2}{x^3}\right)\left(\frac{d^2}{dx^2} - \frac{6}{x^2}\right)\psi_2 = \Phi_1 \tag{6}$$

$$\left(\Phi_1 = \frac{2}{x^2}\psi_1\left(\frac{d}{dx} - \frac{2}{x}\right)\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)\psi_1\right)$$

The boundary conditions are now

$$\psi_1 = \psi_2 = 0, \quad d\psi_1 / dx = d\psi_2 / dx = 0 \ \text{for} \quad x = 1/R \tag{7}$$

$$\psi_1 \to {}^1/_2 R^2x^2 \ \text{for}\ x \to \infty \tag{8}$$

It follows from Eq. (5) that

$$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)\psi_1 = A_1\left[2x^2 - (2x^2 + 2x + 1)\exp\left(-\frac{1}{x}\right)\right] \tag{9}$$

up to an arbitrary constant A_1.

The other linearly independent solution of Eq. (5) is ignored since it leads to $\psi_1 \sim x^4$ for $x \to \infty$.

Equation (9) and the conditions (7) and (8) enable one to obtain

$$\psi_1 = \frac{1}{5}A_1x^4\left[1 + \frac{5(1+R)e^{-R} - 5}{3R^2x^2} + \frac{2 - 2(1+R)e^{-R} - 3R^2E_4(R)}{3R^5x^5} - \exp\left(-\frac{1}{x}\right) - \frac{4}{x}E_5\left(\frac{1}{x}\right)\right] \tag{10}$$

$$\left(A_1 = \frac{3R^4}{2(1+R)e^{-R} - 2 + R^2}, \quad E_n(z) = \int\limits_1^\infty e^{-zt}\frac{dt}{t^n}\right)$$

Hence, it follows that for $x \gg 1$ and $Rx \gg 1$, one has

$$\psi_1 = {}^1/_2\ R^2x^2\ (1 - {}^1/_3 A_1 / R^2x)$$

For low blowing velocities ($R \ll 1$) the above formula is identical with the stream function far away from the sphere of radius $r = 1$

$$\psi_1 = {}^1/_2 R^2x^2\ (1 - 3/2Rx)$$

For intensive blowing ($R \gg 1$) the stream function far away from the sphere is of the same form as for the flow past the sphere of radius $r \sim R\ (x \sim 1)$

$$\psi_1 = {}^1/_2 R^2x^2\ (1 - 1/x)$$

Moreover, the right-hand side of Eq. (6) can be evaluated,

$$\Phi_1 = -\frac{2}{5}A_1^2\left[1 + \frac{5(1+R)e^{-R} - 5}{3R^2x^2} + \frac{2 - 2(1+R)e^{-R} - 3R^2E_4(R)}{3R^5x^5} - \exp\left(-\frac{1}{x}\right) - \frac{4}{x}E_5\left(\frac{1}{x}\right)\right]\exp\left(-\frac{1}{x}\right)$$

It can be shown that a solution of Eq. (6) which for $x \to \infty$ does not increase more rapidly than x^3, is given by the equation

$$\left(\frac{d^2}{dx^2} - \frac{6}{x^2}\right)\psi_2 = \Phi_2 \tag{11}$$

$$\Phi_2 = -2A_1^2x^3\Big\{\Big[\frac{1}{15} + \frac{(1+R)e^{-R}-1}{3R^2x^2}(4x^2+3x+1) + \frac{2(1+R)e^{-R}-2+3R^2E_1(R)}{30R^5x^5} - 24E_1\left(\frac{1}{x}\right) + 24\exp\left(\frac{1}{x}\right)E_1\left(\frac{2}{x}\right) + 18E_2\left(\frac{1}{x}\right) + 3\exp\left(\frac{1}{x}\right)E_2\left(\frac{2}{x}\right) - 12E_3\left(\frac{1}{x}\right) + 6E_4\left(\frac{1}{x}\right) - 2E_6\left(\frac{1}{x}\right)\Big]\exp\left(-\frac{1}{x}\right) - \left[\frac{50}{3} + \frac{4(1+R)e^{-R}-4}{3R^2} - 24\ln 2\right]\left(1-\frac{1}{4x}\right) + A_2\left[\left(24+\frac{18}{x}+\frac{6}{x^2}+\frac{1}{x^3}\right)\exp\left(-\frac{1}{x}\right) - 24 + \frac{6}{x}\right]\Big\}$$

In the above A_2 is an arbitrary constant.

It follows from Eq. (11) and the condition that ψ does not increase more rapidly than x^2 for $x \to \infty$ that

$$\begin{aligned}\psi_2 = -2A_1^2x^5\Big\{&\frac{1}{75}\left[E_3\left(\frac{1}{x}\right) - E_8\left(\frac{1}{x}\right)\right] + \frac{(1+R)e^{-R}-1}{15R^2}\Big[4E_3\left(\frac{1}{x}\right) + \frac{3}{x}E_2\left(\frac{1}{x}\right) + \frac{1}{x^2}E_1\left(\frac{1}{x}\right) - 4E_8\left(\frac{1}{x}\right) - \frac{3}{x}E_7\left(\frac{1}{x}\right) - \frac{1}{x^2}E_6\left(\frac{1}{x}\right)\Big] + \\ &+ \frac{2(1+R)e^{-R}-2+3R^2E_1(R)}{150R^5x^2}\left[\left(\frac{1}{x^2}+\frac{2}{x}+2\right)\exp\left(-\frac{1}{x}\right) - \frac{1}{x^3}E_3\left(\frac{1}{x}\right)\right] - \\ &- \left[\frac{1}{10}E_4\left(\frac{1}{x}\right) + \frac{1}{15}E_3\left(\frac{1}{x}\right) - \frac{5}{7}E_2\left(\frac{1}{x}\right) + \frac{12}{7}E_1\left(\frac{1}{x}\right)\right]\exp\left(-\frac{1}{x}\right) + \\ &+ \frac{1}{2}E_2\left(\frac{2}{x}\right) + \frac{12}{7}E_1\left(\frac{2}{x}\right) + \frac{18x}{175}\left[E_4\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x}\right) - E_4\left(\frac{2}{x}\right)\right] - \\ &- \frac{9}{350}\left[E_4\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x}\right) - E_8\left(\frac{2}{x}\right) - \frac{1}{2x}E_4^2\left(\frac{1}{x}\right)\right] + \\ &+ \frac{38}{525}\left[E_3\left(\frac{2}{x}\right) - E_8\left(\frac{2}{x}\right)\right] + \frac{1}{21x}\left[E_2\left(\frac{2}{x}\right) - E_7\left(\frac{2}{x}\right)\right] - \\ &- \left[\frac{50}{3} + \frac{4(1+R)e^{-R}-4}{3R^2} - 24\ln 2\right]\left(\frac{1}{14} - \frac{1}{24x}\right) + \\ &+ 24A_2\left[E_3\left(\frac{1}{x}\right) - E_8\left(\frac{1}{x}\right)\right] + \\ &+ \frac{18}{x}A_2\left[E_2\left(\frac{1}{x}\right) - E_7\left(\frac{1}{x}\right)\right] + \frac{6}{x^2}A_2\left[E_1\left(\frac{1}{x}\right) - E_6\left(\frac{1}{x}\right)\right] + \\ &+ \frac{1}{x^3}A_2\left[x\exp\left(-\frac{1}{x}\right) - E_5\left(\frac{1}{x}\right)\right] - \frac{12}{7}A_2 + \frac{1}{x}A_2 + \frac{B_2}{x^7} - \\ &- \left[\frac{19}{150} + \frac{2(1+R)e^{-R}-2+3R^2E_4(R)}{75R^5} - \frac{14}{75}\ln 2\right]\frac{1}{x^2}\Big\}\end{aligned} \qquad (12)$$

The constants A_2 and B_2 can be determined from the boundary conditions (7).

For $x \gg 1$ and $Rx \gg 1$ it follows from (12) that

$$\psi_2 = -{}^1\!/_8 A_1R^2x^2 \qquad (13)$$

The solution (12) enables one to state that either for $\varepsilon xR^2 \sim 1$, or $\varepsilon rR \sim 1$, the correction to the flow function $\varepsilon^2\Psi_2$ is a quantity of the same order as $\varepsilon\Psi_1$.

Therefore, to find the outer expansion, one should introduce a radial coordinate, $y = \varepsilon rR$, and seek the stream function in the form

$$\Psi(r,\theta) = {}^1\!/_2(y^2/\varepsilon R^2)\sin^2\theta + \psi(y,\theta)$$

One then obtains for $\psi(y, \theta)$ the Oseen equation

$$\left(\cos\theta\frac{\partial}{\partial y} - \frac{\sin\theta}{y}\frac{\partial}{\partial\theta}\right)D^2\psi = D^4\psi \qquad (14)$$

$$\left(D^2 = \frac{\partial^2}{\partial y^2} + \frac{\sin\theta}{y^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

A corollary of Eq. (14) is

$$D^2\psi = C\left(1+\frac{2}{y}\right)\sin^2\theta\exp\left[-\frac{y(1-\cos\theta)}{2}\right]$$

where C is a constant.

The solution of the above equation is given by

$$\psi = -\cos\theta - 2C(1+\cos\theta)\left\{1 - \exp\left[-\frac{y(1-\cos\theta)}{2}\right]\right\} \quad (15)$$

The function (15) differs from the familiar solution [1–3] in regard to its leading term and is related to the nonzero fluid flow on the sphere surface.

By writing the outer solution (15) in the inner variables and by expanding into series for small εR, one can obtain

$$\Psi = -\cos\theta + {}^1/_2\, R^2x^2\sin^2\theta - C\varepsilon R^2 x \sin^2\theta$$

This solution is now matched with the solution (10) if $C = {}^1/_6 A_1/R^2$.

Thus, the outer solution rewritten by using the inner variables is given by

$$\Psi = \frac{\varepsilon R^2x^2}{2}\sin^2\theta - \frac{A_1}{3R^2}(1+\cos\theta)\left\{1 - \exp\left[-\frac{\varepsilon R^2 x}{2}(1-\cos\theta)\right]\right\} - \cos\theta \quad (16)$$

The ratio of the second to the leading term of the series, for a fixed value of the outer variable $\varepsilon R^2 x$, is equal in regard to the order of the quantities to εR for $R \ll 1$, or to εR^2 for $R \gg 1$.

The outer solution (16) can be matched with an inner one which with an error of up to the order of ε^2 inclusive should be selected in the form

$$\Psi = -\cos\theta + \varepsilon\,(1 + {}^1/_{12}\varepsilon A_1)\,\psi_1(x)\sin^2\theta + \varepsilon^2\psi_2(x)\sin^2\theta\cos\theta$$

where ψ_1 and ψ_2 are described by the formulas (10) or (12), respectively.

The force acting on the sphere is determined by the momentum flow on its surface. The projection of the force onto the velocity direction of the oncoming flow **u** is given by

$$Q = 2\pi a^2\int_0^\pi [(-\rho w^2 v_r^2 - p + \sigma_{rr}')\cos\theta - \sigma_{r\theta}'\sin\theta]\sin\theta\, d\theta$$
$$(R\sigma_{rr}' = 2\rho w^2\partial v_r/\partial r, \quad R\sigma_{r\theta}' = \rho w^2(r^{-1}\partial v_r/\partial\theta + \partial v_\theta/\partial r - v_\theta/r)) \quad (17)$$

For a velocity distribution which is spherically symmetric the convective momentum transfer makes no direct contribution to the integral.

To evaluate the resistive force **Q** it is sufficient to consider only those terms in pressure p which are proportional to $\cos\theta$. Other terms can be ignored either in view of antisymmetry of the function under the integral sign in (17) in regard to the change from θ to $\pi-\theta$, or as small quantities of the order of ε^3.

By using the Navier–Stokes equation, one can obtain

$$\frac{p}{\rho w^2} = -\frac{2\varepsilon A}{R^4}\left[1 - \frac{2}{R} + \left(1 + \frac{2}{R}\right)e^{-R}\right]\cos\theta + \ldots$$
$$A = A_1(1 + {}^1/_{12}\,\varepsilon A_1)$$

with accuracy up to the terms of order ε^2, where the dots refer to terms which are not essential for the computation of Q.

Similarly, when evaluating σ_{rr}' and $\sigma_{r\theta}'$ it only suffices to consider the terms

$$\sigma_{rr}' = 0, \quad \frac{\sigma_{r\theta}'}{\rho w^2} = -\frac{2\varepsilon A}{R^5}\left[1 - \left(1 + R + \frac{R^2}{2}\right)e^{-R}\right]\sin\theta$$

Thus

$$Q = \frac{8\pi\rho\nu R a u}{2(1+R)e^{-R} - 2 + R^2}[1 - (1+R)e^{-R}]\left[1 + \frac{\varepsilon}{4}\frac{R^4}{2(1+R)e^{-R} - 2 + R^2}\right] \quad (18)$$

The first step of this series in powers of ε was obtained previously in [5].

For low velocity of blowing ($R \ll 1$) the resistance force is identical with the Oseen formula,

$$Q = 6\pi\rho\nu a u (1 + {}^3/_8 \varepsilon R) \tag{19}$$

For intensive blowing, one has

$$Q = (8\pi / R) \rho\nu a u (1 + {}^1/_4 \varepsilon R^2) \tag{20}$$

The region of applicability of the formula corresponds to $\varepsilon R^2 \ll 1$ which is equivalent to the requirement of low Reynolds numbers evaluated from the velocity of the flow past a sphere of effective radius $r \sim R$. In that region the resistive force may be considerably lower than the one given by the Stokes formula.

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